

THE COMPLEX MONGE-AMPÈRE EQUATION ON WEAKLY PSEUDOCONVEX DOMAINS

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ABSTRACT. We show here a “weak” Hölder-regularity up to the boundary of the solution to the Dirichlet problem for the complex Monge-Ampère equation with data in the L^p space and Ω satisfying an f -property. The f -property is a potential-theoretical condition which holds for all pseudoconvex domains of finite type and many examples of infinite type.

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1. INTRODUCTION

For a C^2 , bounded, pseudoconvex domain $\Omega \subset \subset \mathbb{C}^n$, the Dirichlet problem for the Monge-Ampère equation consists in

$$\begin{cases} u \in PSH(\Omega) \cap L_{loc}^\infty(\Omega), \\ (dd^c u)^n = \psi dV & \text{in } \Omega, \\ u = \varphi & \text{on } b\Omega. \end{cases} \quad (1.1)$$

A great deal of work has been done when Ω is strongly pseudoconvex. On this domain, we can divide the literature by three kinds of data ψ .

- The Hölder data: Bedford-Taylor prove in [2] that $u \in C^{\frac{\alpha}{2}}(\bar{\Omega})$ if $\varphi \in C^\alpha(b\Omega)$, $\psi^{\frac{1}{n}} \in C^{\frac{\alpha}{2}}(\bar{\Omega})$ for $0 < \alpha \leq 2$.
- The smooth data: Caffarelli, Kohn and Nirenberg prove in [4] that $u \in C^\infty(\bar{\Omega})$, for $\varphi \in C^\infty(b\Omega)$ and $\psi \in C^\infty(\bar{\Omega})$, in case $\psi > 0$ in $\bar{\Omega}$ and $b\Omega$ is smooth.
- The L^p data: Guedj, Kolodziej and Zeriahi prove in [6] that if $\psi \in L^p(\Omega)$ with $p > 1$ and $\varphi \in C^{1,1}(b\Omega)$ then $u \in C^\gamma(\bar{\Omega})$ for any $\gamma < \gamma_p := \frac{2}{qn+1}$ where $\frac{1}{q} + \frac{1}{p} = 1$.

When Ω is no longer strongly pseudoconvex but has a certain “finite type”, there are some known results for this problem due to Blocki [3], Coman [5], and Li [11]. Recently, Ha and the second author gave a general related result to a Hölder data under the hypothesis that Ω satisfies an f -property (see Definition 2.1 below). The f -property is a consequence of the geometric “type” of the boundary. All pseudoconvex domains of finite type satisfy the f -property as well as many classes of domains of infinite type (see [7, 8, 9] for discussion on the f -property). Using the f -property, a “weak” Hölder regularity for the solution of the Dirichlet problem of complex Monge-Ampère equation is obtained in [7]. Coming back to the case of Ω of finite type, in a recent paper with Zampieri [1], we prove the Hölder regularity for $\psi \in L^p$ with $p > 1$. The purpose of the present paper is to generalize the result in [1] to a pseudoconvex domain satisfying an f -property. For this purpose we recall the definition of a weak Hölder space in [7, 8]. Let f be an increasing function such that $\lim_{t \rightarrow +\infty} f(t) = +\infty$,

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$f(t) \lesssim t$. For a subset A of \mathbb{C}^n , define the f -Hölder space on A by

$$\Lambda^f(A) = \{u : \|u\|_{L^\infty(A)} + \sup_{z, w \in A, z \neq w} f(|z - w|^{-1}) \cdot |u(z) - u(w)| < \infty\}$$

and set

$$\|u\|_{\Lambda^f(A)} = \|u\|_{L^\infty(A)} + \sup_{z, w \in A, z \neq w} f(|z - w|^{-1}) \cdot |u(z) - u(w)|.$$

Note that the notion of the f -Hölder space includes the standard Hölder space Λ_α by taking $f(t) = t^\alpha$ (so that $f(|h|^{-1}) = |h|^{-\alpha}$) with $0 < \alpha \leq 1$. Here is our result

Theorem 1.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded, pseudoconvex domain admitting the f -property. Suppose that $\int_1^\infty \frac{da}{af(a)} < \infty$ and denote by $g(t) := \left(\int_t^\infty \frac{da}{af(a)}\right)^{-1}$ for $t \geq 1$. If $0 < \alpha \leq 2$, $\varphi \in \Lambda^{t^\alpha}(b\Omega)$, and $\psi \geq 0$ on Ω with $\psi \in L^p$ with $p > 1$, then the Dirichlet problem for the complex Monge-Ampère equation (1.1) has a unique plurisubharmonic solution $u \in \Lambda^{g^\beta}(\bar{\Omega})$. Here $\beta = \min(\alpha, \gamma)$, for any $\gamma < \gamma_p = \frac{2}{nq+1}$ where $\frac{1}{p} + \frac{1}{q} = 1$.*

The proof follows immediately from Theorem 2.2 and 2.5 below. Throughout the paper we use \lesssim and \gtrsim to denote an estimate up to a positive constant, and \approx when both of them hold simultaneously. Finally, the indices p, α, β, γ and γ_p only take ranges as in Theorem 1.1.

2. HÖLDER REGULARITY OF THE SOLUTION

We start this section by defining the f -property as in [8, 9].

Definition 2.1. For a smooth, monotonic, increasing function $f : [1, +\infty) \rightarrow [1, +\infty)$ with $f(t)t^{-1/2}$ decreasing, we say that Ω has the f -property if there exist a neighborhood U of $b\Omega$ and a family of functions $\{\varphi_\delta\}$ such that

- (i) the functions φ_δ are plurisubharmonic, C^2 on U , and satisfy $-1 \leq \varphi_\delta \leq 0$,
- (ii) $i\partial\bar{\partial}\varphi_\delta \gtrsim f(\delta^{-1})^2 Id$ and $|D\varphi_\delta| \lesssim \delta^{-1}$ for any $z \in U \cap \{z \in \Omega : -\delta < r(z) < 0\}$, where r is a C^2 -defining function of Ω .

In [8], using the f -property, the second author constructed a family of plurisubharmonic peak functions with good estimates. This family of plurisubharmonic peak functions yields the existence of a defining function ρ which is uniformly strictly-plurisubharmonic and weakly Hölder (see [7]).

Theorem 2.2 (Khanh [8] and Ha-Khanh[7]). *Assume that Ω is a bounded, pseudoconvex domain admitting the f -property as in Theorem 1.1. Then there exists a uniformly strictly-plurisubharmonic defining function of Ω which belongs to the g^2 -Hölder space of $\bar{\Omega}$, that means,*

$$\rho \in \Lambda^{g^2}(\bar{\Omega}), \quad \Omega = \{\rho < 0\} \quad \text{and} \quad i\partial\bar{\partial}\rho \geq Id. \quad (2.1)$$

The existence and uniqueness of the solution $u \in L^\infty(\Omega)$ to the equation (1.1) need a weaker condition, in particular, one only need $\rho \in C^0(\bar{\Omega})$ as shown by [10].

Theorem 2.3 (Kolodziej [10]). *Let Ω be a bounded domain in \mathbb{C}^n . Assume that there exists a function ρ such that*

$$\rho \in C^0(\bar{\Omega}), \quad \Omega = \{\rho < 0\} \quad \text{and} \quad i\partial\bar{\partial}\rho \geq Id.$$

Then for any $\varphi \in C^0(b\Omega)$, $\psi \in L^p(\Omega)$ there is a unique plurisubharmonic solution $u(\Omega, \varphi, \psi) \in C^0(\bar{\Omega})$.

To improve the smoothness of u , we increase the smoothness of ρ and ψ .

Theorem 2.4 (Ha-Khanh [7]). *Let ρ satisfy (2.1). If $\varphi \in \Lambda^{t^\alpha}(b\Omega)$ and $\psi^{\frac{1}{n}} \in \Lambda^{g^\alpha}(\bar{\Omega})$, then the Dirichlet problem for the complex Monge-Ampère equation (1.1) has a unique plurisubharmonic solution $u(\Omega, \varphi, \psi) \in \Lambda^{g^\alpha}(\bar{\Omega})$.*

Now we focus on lowering the smoothness of ψ and prove the following theorem.

Theorem 2.5. *Let ρ satisfy (2.1). If $\varphi \in \Lambda^{t^\alpha}(b\Omega)$ and $\psi \in L^p(\Omega)$, then the Dirichlet problem for the complex Monge-Ampère equation (1.1) has a unique plurisubharmonic solution $u(\Omega, \varphi, \psi) \in \Lambda^{g^\beta}(\bar{\Omega})$.*

In order to prove this theorem, we need to construct a subsolution with L^p data. Here, v is a subsolution to (1.1) in the sense that v is plurisubharmonic, $v|_{b\Omega} = \varphi$ and $(dd^c v)^n \geq \psi dV$ in Ω .

Proposition 2.6. *Let ρ satisfy (2.1). Then there is a subsolution $v \in \Lambda^{g^\beta}(\bar{\Omega})$ to (1.1) for $\varphi \in C^\alpha(b\Omega)$ and $\psi \in L^p(\Omega)$.*

Proof. For a large ball \mathbb{B} containing Ω , we set $\tilde{\psi}(z) := \begin{cases} \psi(z) & \text{if } z \in \Omega, \\ 0 & \text{if } z \in \mathbb{B} \setminus \Omega. \end{cases}$ First, we

apply Theorem 1 in [6] on \mathbb{B} with $\tilde{\psi} \in L^p(\mathbb{B})$ and zero-valued boundary condition, it follows $u_1 = u(\mathbb{B}, 0, \tilde{\psi}) \in \Lambda^{t^\gamma}(\bar{\mathbb{B}})$. Second, we apply Theorem 2.4 on Ω twice: first for $u_2 := u(\Omega, -u_1|_{b\Omega}, 0) \in \Lambda^{g^\gamma}$ since $u_1|_{b\Omega} \in \Lambda^{t^\gamma}$ and second for $u_3 := u(\Omega, \varphi, 0) \in \Lambda^{g^\alpha}$ by the hypothesis $\varphi \in \Lambda^{t^\alpha}$. Finally, taking the summation $v = u_1 + u_2 + u_3$, we have the conclusion. \square

Proof of Theorem 2.5. Keeping the notation of Theorem 2.3, let $u(\Omega, \varphi, \psi) \in C^0(\bar{\Omega})$ be the solution of (1.1). What follows is dedicated to showing that this C^0 plurisubharmonic solution $u(\Omega, \varphi, \psi)$ is in fact in $\Lambda^{g^\beta}(\bar{\Omega})$. By Theorem 2.4, $w := u(\Omega, \varphi, 0) \in \Lambda^{g^\alpha}(\bar{\Omega})$; let v be as in Proposition 2.6, comparison principle yields at once

$$v \leq u(\Omega, \varphi, \psi) \leq w. \quad (2.2)$$

By (2.2) and the g^β -Hölder regularity of v and w we get

$$|u(z) - u(\zeta)| \lesssim [g(|z - \zeta|^{-1})]^{-\beta} \quad z \in \bar{\Omega}, \quad \zeta \in b\Omega,$$

and therefore for δ suitably small

$$|u(z) - u(z')| \lesssim [g(\delta^{-1})]^{-\beta}, \quad z, z' \in \Omega \setminus \Omega_\delta \text{ and } |z - z'| < \delta \quad (2.3)$$

where $\Omega_\delta := \{z \in \mathbb{C}^n : r(z) < -\delta\}$ and r is the C^2 defining function for Ω with $|\nabla r| = 1$ on $b\Omega$. We have to prove that (2.3) also holds for $z, z' \in \Omega_\delta$. For $z \in \bar{\Omega}_\delta$, we use the notation

$$u_{\frac{\delta}{2}}(z) := \sup_{|\zeta| < \frac{\delta}{2}} u(z + \zeta), \quad \tilde{u}_{\frac{\delta}{2}}(z) := \frac{1}{\sigma_{2n-1}(\frac{\delta}{2})^{2n-1}} \int_{b\mathbb{B}(z, \frac{\delta}{2})} u(\zeta) dS(\zeta),$$

and

$$\hat{u}_{\frac{\delta}{2}}(z) := \frac{1}{\sigma_{2n}(\frac{\delta}{2})^{2n}} \int_{\mathbb{B}(z, \frac{\delta}{2})} u(\zeta) dV(\zeta),$$

where $\sigma_{2n-1}(\frac{\delta}{2})^{2n-1} = \text{Vol}(b\mathbb{B}(z, \frac{\delta}{2}))$ and $\sigma_{2n}(\frac{\delta}{2})^{2n} = \text{Vol}(\mathbb{B}(z, \frac{\delta}{2}))$. It is obvious that

$$\hat{u}_{\frac{\delta}{2}} \leq \tilde{u}_{\frac{\delta}{2}} \leq u_{\frac{\delta}{2}} \quad \text{in } \Omega_{\delta}. \quad (2.4)$$

Furthermore, we have L^1 estimate of the difference between u and $\tilde{u}_{\frac{\delta}{2}}$ and the stability estimate in the following theorems:

Theorem 2.7 (Baracco-Khanh-Pinton-Zampieri [1]). *For any $0 < \epsilon < 1$, we have*

$$\|\tilde{u}_{\frac{\delta}{2}} - u\|_{L^1(\Omega_{\delta})} \lesssim \delta^{1-\epsilon}. \quad (2.5)$$

Theorem 2.8 (Guedj-Kolodziej-Zeriahi [6]). *Fix $0 \leq f \in L^p(\Omega)$, $p > 1$. Let U, W be two bounded plurisubharmonic functions in Ω such that $(dd^c U)^n = f dV$ in Ω and let $U \geq W$ on $\partial\Omega$. Fix $s \geq 1$ and $0 \leq \eta < \frac{s}{nq+s}$, $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists a uniform constant $C = C(\eta, \|f\|_{L^p(\Omega)}) > 0$ such that*

$$\sup_{\Omega} (W - U) \leq C \|(W - U)_+\|_{L^s(\Omega)}^{\eta},$$

where $(W - U)_+ := \max(W - U, 0)$.

By (2.3), we have

$$\tilde{u}_{\frac{\delta}{2}} \leq u_{\frac{\delta}{2}} \leq u + c[g(\delta^{-1})]^{-\beta}, \quad \text{on } b\Omega_{\delta} \text{ for suitable constant } c.$$

Thus, we can apply Theorem 2.8 for Ω_{δ} with $U := u + c[g(\delta^{-1})]^{-\beta}$, $W := \tilde{u}_{\frac{\delta}{2}}$ and $s := 1$; thus we get

$$\begin{aligned} \sup_{\Omega_{\delta}} \left(\tilde{u}_{\frac{\delta}{2}} - (u + c[g(\delta^{-1})]^{-\beta}) \right) &\stackrel{\text{Theorem 2.8}}{\lesssim} \left\| \left(\tilde{u}_{\frac{\delta}{2}} - (u + c[g(\delta^{-1})]^{-\beta}) \right)_+ \right\|_{L^1(\Omega_{\delta})}^{\eta} \\ &\lesssim \|\tilde{u}_{\frac{\delta}{2}} - u\|_{L^1(\Omega_{\delta})}^{\eta} \stackrel{\text{Theorem 2.7}}{\lesssim} \delta^{(1-\epsilon)\eta}, \end{aligned} \quad (2.6)$$

for any $\eta < \frac{1}{2}\gamma_p = \frac{1}{nq+1}$ where $\frac{1}{q} + \frac{1}{p} = 1$. Taking $\gamma < \gamma_p$, $\beta = \min(\alpha, \gamma)$, $\epsilon = \frac{\gamma_p - \gamma}{\gamma_p + \gamma} > 0$ and $\eta = \frac{1}{4}(\gamma_p + \gamma) < \frac{1}{2}\gamma_p$ so that $(1 - \epsilon)\eta = \frac{\gamma}{2}$, it follows

$$\sup_{\Omega_{\delta}} \left(\tilde{u}_{\frac{\delta}{2}} - u \right) \lesssim \delta^{(1-\epsilon)\eta} + [g(\delta^{-1})]^{-\beta} \lesssim \delta^{\frac{\gamma}{2}} + [g(\delta^{-1})]^{-\beta} \lesssim [g(\delta^{-1})]^{-\beta}, \quad (2.7)$$

where the last inequality of (2.7) follows by $g(\delta^{-1}) \lesssim \delta^{-\frac{1}{2}}$ (by the conditions on f in the f -property).

Similarly to [6, Lemma 4.2] by using the fact that $g(c\delta^{-1}) \approx g(\delta^{-1})$ for any constant $c > 0$, one can state the equivalence between

$$\sup_{\Omega_{\delta}} (u_{\delta} - u) \lesssim [g(\delta^{-1})]^{-\beta} \quad \text{and} \quad \sup_{\Omega_{\delta}} (\hat{u}_{\delta} - u) \lesssim [g(\delta^{-1})]^{-\beta}.$$

Using this equivalence together with the inequalities in (2.4), it follows that (2.7) is equivalent to

$$\sup_{\Omega_{\delta}} (u_{\frac{\delta}{2}} - u) \lesssim [g(\delta^{-1})]^{-\beta}. \quad (2.8)$$

From (2.3) and (2.8), it is easy to prove that

$$|u(z) - u(z')| \lesssim [g(|z - z'|^{-1})]^{-\beta} \quad \text{for any } z, z' \in \bar{\Omega}.$$

□

REFERENCES

- [1] Luca Baracco, Tran Vu Khanh, Stefano Pinton, and Giuseppe Zampieri. Hölder regularity of the solution to the complex Monge-Ampère equation with L^p density. *Calc. Var. Partial Differential Equations*, 55(4):55:74, 2016.
- [2] Eric Bedford and B. A. Taylor. The Dirichlet problem for a complex Monge-Ampère equation. *Invent. Math.*, 37(1):1–44, 1976.
- [3] Zbigniew Blocki. The complex Monge-Ampère operator in hyperconvex domains. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 23(4):721–747 (1997), 1996.
- [4] L. Caffarelli, J. J. Kohn, L. Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic, equations. *Comm. Pure Appl. Math.*, 38(2):209–252, 1985.
- [5] Dan Coman. Domains of finite type and Hölder continuity of the Perron-Bremermann function. *Proc. Amer. Math. Soc.*, 125(12):3569–3574, 1997.
- [6] Vincent Guedj, Slawomir Kolodziej, and Ahmed Zeriahi. Hölder continuous solutions to Monge-Ampère equations. *Bull. Lond. Math. Soc.*, 40(6):1070–1080, 2008.
- [7] Ly Kim Ha and Tran Vu Khanh. Boundary regularity of the solution to the complex Monge-Ampère equation on pseudoconvex domains of infinite type. *Math. Res. Lett.*, 22(2):467–484, 2015.
- [8] Tran Vu Khanh. Lower bounds on the Kobayashi metric near a point of infinite type. *J. Geom. Anal.*, 26(1):616–629, 2016.
- [9] Tran Vu Khanh and Giuseppe Zampieri. Regularity of the $\bar{\partial}$ -Neumann problem at point of infinite type. *J. Funct. Anal.*, 259(11):2760–2775, 2010.
- [10] Slawomir Kolodziej. The complex Monge-Ampère equation. *Acta Math.*, 180(1):69–117, 1998.
- [11] Song-Ying Li. On the existence and regularity of Dirichlet problem for complex Monge-Ampère equations on weakly pseudoconvex domains. *Calc. Var. Partial Differential Equations*, 20(2):119–132, 2004.

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